PARAMETRIC ESTIMATION OF UNIFORM EFFECT WITH NORMAL ERROR

by

YUEN WAH-KONG

A

Thesis

Submitted to

the Graduate School of

The Chinese University of Hong Kong

(Division of Mathematics)

In Partial Fulfillment

of the Requirements for the Degree of

Master of Philosophy (M. Phil.)

HONG KONG

June 1980

THE CHINESE UNIVERSITY OF HONG KONG

GRADUATE SCHOOL

The undersigned certify that we have read a thesis, entitled "Parametric Estimation of Uniform Effect with Normal Error" submitted to the Graduate School by Mr. Yuen Wah-Kong (阮華剛) is partial fulfillment of the requirement for the degree of Master of Philosophy in Mathematics. We recommend that it be accepted.

(Dr. N.N. Chan) Supervisor

(Dr. H.K. Lam)

(Dr. S.Y. Lee)

(Prof. J. Aitchison)

ACKNOWLEDGEMENT

I would like to express my hearty gratitude to my supervisor Dr N.N. Chan for his encouragement and criticism during this work. I would also like to thank Mr Billy Lam for his typing.

PARAMETRIC ESTIMATION OF UNIFORM EFFECT WITH NORMAL ERROR

Chapter 1. Introduction

The purpose of this thesis is to treat an estimation problem which arises when one observes a sum of two random variables $Z = X + Y$, where X and Y are independent, and not observable individually. Throughout this thesis, it is assumed that X is normally distributed with mean zero and variance σ^2, i.e., $X \sim N(0, \sigma^2)$, and Y is uniformly distributed in the interval $(\mu - \tau, \mu + \tau)$, i.e., $Y \sim U(\mu - \tau, \mu + \tau)$. The question at hand is that of estimating the parameters μ, τ, σ^2, given observations on Z only.

Denote by ϕ and Φ the density function and the distribution function, respectively, of the standard normal random variable, i.e.,

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

and

$$\Phi(x) = \int_{-\infty}^{x} \phi(u)\,du .$$

The density function of $Z = X + Y$ is then

$$f(z) = \frac{1}{2\tau}\int_{\mu-\tau}^{\mu+\tau} \frac{1}{\sigma}\phi\left(\frac{z-t}{\sigma}\right) d\,t$$

$$= \frac{1}{2\tau}\left[\Phi\left(\frac{z-\mu+\tau}{\sigma}\right) - \Phi\left(\frac{z-\mu-\tau}{\sigma}\right)\right], \quad \ldots\ldots\ldots \qquad (1.1)$$

the range of z being $(-\infty, \infty)$. The mean and variance of Z are

$$E(Z) = E(X + Y) = E(X) + E(Y) = 0 + \mu = \mu$$

$$\mathrm{var}(Z) = \mathrm{var}(X + Y) = \mathrm{var}(X) + \mathrm{var}(Y) = \sigma^2 + \tau^2/3$$

Write $M_Z(t) = E(e^{tZ})$ which is the moment generating function of Z. Since

$$E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx$$

$$= e^{\frac{\sigma^2 t^2}{2}} ,$$

and

$$E(e^{tY}) = \int_{\mu-\tau}^{\mu+\tau} e^{ty} \frac{1}{2\tau} dy$$

$$= \frac{1}{2t\tau} [e^{t(\mu+\tau)} - e^{t(\mu-\tau)}]$$

$$= \frac{1}{2t\tau} e^{t\mu} (e^{t\tau} - e^{-t\tau}) ,$$

we have

$$M_Z(t) = e^{\frac{\sigma^2 t^2}{2}} e^{t\mu} \frac{\sinh t\tau}{t\tau} \quad \ldots\ldots\ldots\ldots \quad (1.2)$$

The characteristic function of Z, denoted by $\hat{f}(t)$, is similarly found to be

$$\hat{f}(t) = e^{-\frac{\sigma^2 t^2}{2}} \frac{1}{2i\,t\tau} e^{it\mu} (e^{it\tau} - e^{-it\tau})$$

$$= e^{-\frac{\sigma^2 t^2}{2}} e^{it\mu} \frac{\sin t\tau}{t\tau} \quad \ldots\ldots\ldots\ldots \quad (1.3)$$

Historically, the density function of Z is firstly presented in Bhattacharjee (1963). In Chapter 2 and 3, we shall discuss maximum likelihood estimation and estimation by method of moments, respectively. In Chapter 4, a different approach is attempted which is originally suggested by Press (1972) who applied it to parametric estimation of the stable distributions. He estimated the parameters by minimizing an appropriate 'distance' between the characteristic function of the random variable and the sample characteristic function of the observations. Several Monte-Carlo simulation studies of the estimates obtained are presented in Chapter 5.

Chapter 2. Maximum Likelihood Estimation and Allied Methods

In this chapter the problem of estimating the parameters in the distribution of Z based on the likelihood function is considered using various approaches. In section 2.1 the method of maximum likelihood and a combined method which is composed of solving some maximum likelihood equations and moment equations are discussed. The general theory and method of 'correct' likelihood proposed by Giesbrecht and Kempthorne (1976) is discussed in section 2.2. The estimation procedure for the above methods is followed in section 2.3. The behaviour of the above estimation procedure is discussed in section 2.4.

§2.1 Method of Maximum Likelihood and the Combined Method

Consider n independent observations $z_1, \ldots, z_n$ from the distribution (1.1) of chapter 1. The likelihood function is then

$$L(z_1, \ldots, z_n \mid \mu, \tau, \sigma)$$

$$= \frac{1}{(2\tau)^n} \prod_{i=1}^{n} \left[\Phi\left(\frac{z_i - \mu + \tau}{\sigma}\right) - \Phi\left(\frac{z_i - \mu - \tau}{\sigma}\right) \right] \quad \ldots\ldots\ldots \quad (2.1)$$

We are interested in the estimation of the parameters μ, τ, σ using (2.1).

2.1.1 Maximum Likelihood Method

According to the usual optimization procedure, a necessary condition for the likelihood function to attain its maximum is

$$\frac{\partial \ln L}{\partial \mu} = \frac{\partial \ln L}{\partial \tau} = \frac{\partial \ln L}{\partial \sigma} = 0 \quad \ldots\ldots\ldots\ldots \quad (2.2)$$

An estimation procedure for the method by solving the equations (2.2) is given in section 2.3.1.

Generally, explicit expressions for the estimators arising as solutions of (2.2) are not available, i.e., we cannot explicitly express μ, τ, σ as functions of $z_1, \ldots, z_n$, and numerical methods are usually used to obtain the estimates. In some cases the behaviour of the iterative formulae may not be satisfactory, e.g., the iterative process may be divergent or have very slow convergence rate.

2.1.2 Combined Method

In view of the defect of the iterative procedure discussed above, the estimation procedure may sometimes be carried out as follows: Some of the iterative formulae derived from (2.2) which are convergent and with fast convergence rate together with r equations from the method of moments, i.e.,

$$\mu_i = f_i(z_1, \ldots, z_n) \qquad \ldots\ldots\ldots\ldots\ldots \qquad (2.3)$$

$i = 1, \ldots, r$, $r < 3$, here μ_i is the i^{th} central moment, f_i is the i^{th} sample central moment, μ_i is function of μ, τ, σ respectively, are used for estimation. The total number of equations is confined to three, i.e., the number of parameters. Under the condition that some of the equations of (2.3) can be solved for certain parameters, the estimation of the remaining parameters is carried out by iteration and substitution between remaining formulae. All the previous discussion will be demonstrated in section 2.3.2.

§2.2. 'Correct' likelihood

Several statisticians, including Fisher (1921), Kempthorne (1966), Barnard (1967) and Lambert (1970) claimed that all previous work performed as though the observations had recorded without error. They argued that all observations were subject to a grouping error, and hence were recorded discrete. It followed that the actual observations is in some interval $(x - \frac{1}{2}\Delta\ , x + \frac{1}{2}\Delta\ )$, where Δ is a real constant. We consider the probability of the actual data as a function of the unknown parameters. This function is called the "correct" likelihood by Giesbrecht and Kempthorne (1976). Also, the powerful theorems in Kulldorff (1957) concerning maximum likelihood estimation in the case of the observations from a multinomial distribution can be applied.

Let L_c be the 'correct' likelihood function, i.e.,

$$L_c(x_1, \ldots, x_n \mid \alpha_1, \ldots, \alpha_k) = K \prod_i p_i^{f_i} \quad \ldots\ldots\ldots \qquad (2.4)$$

where K is a multinomial coefficient, f_i is the number of observations recorded between $(i - \frac{1}{2})\Delta$ and $(i + \frac{1}{2})\Delta$, here Δ is a fixed constant, and the $\{p_i\}$ are defined by

$$p_i = \int_{(i - \frac{1}{2})\Delta}^{(i + \frac{1}{2})\Delta} f(x)dx \ .$$

Note that all grouping intervals have equal length. It is conceivable that a more complicated but realistic structure would allow the length of the grouping intervals to depend on the magnitude of the observations so that each interval will have similar number of observations. Because of the complexity of this approach, it will not be pursued here.

Clearly, $p_i \leq 1$ for all i and $L_c(x_1, \ldots, x_n \mid \alpha_1, \ldots, \alpha_n)$

defined by (2.4) is bounded and can be maximized for any fixed Δ . The 'correct' maximum likelihood estimates of the unknown parameters are defined as the set of values that provide the absolute maximum of the 'correct' likelihood. However, there is a positive probability that no such set of values exist, even though this probability may tend to zero as the sample size increases indefinitely. Kulldorff (1957) gives sufficient conditions for these estimates to be consistent and assymptotically efficient provided that the 'correct' maximum likelihood estimates exists. This discussion will not be continued in this thesis.

§2.3. The Estimation Procedure

2.3.1 Maximum likelihood estimation

The likelihood function is

$$L(z_1, \ldots, z_n|\mu, \tau, \sigma) = \prod_{i=1}^{n} \frac{1}{2\tau} [\Phi(\frac{z_i - \mu + \tau}{\sigma}) - \Phi(\frac{z_i - \mu - \tau}{\sigma})]$$

$$\ln L = -n \ln 2\tau + \sum_{i=1}^{n} \ln[\Phi(\frac{z_i - \mu + \tau}{\sigma}) - \Phi(\frac{z_i - \mu - \tau}{\sigma})]$$

$$\frac{\partial \ln L}{\partial \tau} = -\frac{n}{\tau} + \sum_{i=1}^{n} \frac{\frac{1}{\sigma}\phi(\frac{z_i - \mu + \tau}{\sigma}) + \frac{1}{\sigma}\phi(\frac{z_i - \mu - \tau}{\sigma})}{\Phi(\frac{z_i - \mu + \tau}{\sigma}) - \Phi(\frac{z_i - \mu - \tau}{\sigma})}$$

$$\frac{\partial \ln L}{\partial \sigma} = -\frac{1}{\sigma^2} \sum_{i=1}^{n} \frac{(z_i - \mu + \tau)\phi(\frac{z_i - \mu + \tau}{\sigma}) - (z_i - \mu - \tau)\phi(\frac{z_i - \mu - \tau}{\sigma})}{\Phi(\frac{z_i - \mu + \tau}{\sigma}) - \Phi(\frac{z_i - \mu - \tau}{\sigma})}$$

$$\frac{\partial \ln L}{\partial \mu} = -\frac{1}{\sigma^2} \sum_{i=1}^{n} \frac{\phi(\frac{z_i - \mu + \tau}{\sigma}) - \phi(\frac{z_i - \mu - \tau}{\sigma})}{\Phi(\frac{z_i - \mu + \tau}{\sigma}) + \Phi(\frac{z_i - \mu - \tau}{\sigma})}$$

Let

$$\psi_i = \frac{\phi\left(\frac{z_i - \mu + \tau}{\sigma}\right)}{\Phi\left(\frac{z_i - \mu + \tau}{\sigma}\right) - \Phi\left(\frac{z_i - \mu - \tau}{\sigma}\right)}$$

$$\theta_i = \frac{\phi\left(\frac{z_i - \mu - \tau}{\sigma}\right)}{\Phi\left(\frac{z_i - \mu + \tau}{\sigma}\right) - \Phi\left(\frac{z_i - \mu - \tau}{\sigma}\right)}$$

hence we have

$$\frac{\partial \ln L}{\partial \mu} = -\frac{1}{\sigma^2} \sum_{i=1}^{n} (\psi_i - \theta_i) = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (2.5)$$

$$\frac{\partial \ln L}{\partial \tau} = -\frac{n}{\tau} + \frac{1}{\sigma} \sum_{i=1}^{n} (\psi_i + \theta_i) = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (2.6)$$

$$\frac{\partial \ln L}{\partial \sigma} = -\frac{1}{\sigma^2} \sum_{i=1}^{n} [(z_i - \mu + \tau)\psi_i - (z_i - \mu - \tau)\theta_i] = 0 \quad \ldots\ldots \quad (2.7)$$

We wish to find μ, σ, τ such that the above three equations are satisfied simultaneously.

Since we cannot derive an iterative formula for μ directly from (2.5), we use Newton-Raphson method instead. Let

$$F(\mu, \tau, \sigma) = \sum_{i=1}^{n} (\psi_i - \theta_i) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2.8)$$

$$\frac{\partial F}{\partial \mu} = \frac{1}{\sigma} \sum_{i=1}^{n} \left[\left(\frac{z_i - \mu + \tau}{\sigma}\right)\psi_i - \left(\frac{z_i - \mu - \tau}{\sigma}\right)\theta_i + (\psi_i - \theta_i)^2 \right] \ldots \quad (2.9)$$

We have

$$\mu_{k+1} = \mu_k - \frac{F}{\left.\frac{\partial F}{\partial \mu}\right|_{\mu=\mu_k}} \qquad \ldots\ldots\ldots\ldots\ldots \qquad (2.10)$$

By (2.6) and (2.7), we have

$$\sigma_{k+1} = \frac{\tau_k}{n} \sum_{i=1}^{n} (\psi_i + \theta_i) \qquad \ldots\ldots\ldots\ldots\ldots \qquad (2.11)$$

and

$$\tau_{k+1} = \frac{\sum_{i=1}^{n} (z_i - \mu_k)(\theta_i - \psi_i)}{\sum_{i=1}^{n} (\theta_i + \psi_i)} \qquad \ldots\ldots\ldots\ldots \qquad (2.12)$$

Here all parameters μ, τ, σ in the expressions θ_i and ψ_i are substituted by μ_k, τ_k, σ_k respectively.

2.3.2 The Combined Method

From computational result, the behaviour of equation (2.11) is satisfactory. Here we suggest the following three equations to estimate the three parameters μ, τ, σ^2

$$\hat{\mu} = \sum_{i=1}^{n} Z_{i/n} = \overline{Z} \qquad \ldots\ldots\ldots\ldots\ldots \qquad (2.13)$$

$$\hat{\sigma}^2 + \frac{\hat{\tau}^2}{3} = \frac{\sum_{i=1}^{n} (Z_i - \overline{Z})^2}{n-1} = M_2$$

then

$$\hat{\tau} = \sqrt{3(M_2 - \hat{\sigma}^2)} \qquad \ldots\ldots\ldots\ldots\ldots \qquad (2.14)$$

$$\sigma = \frac{\tau}{n} \sum_{i=1}^{n} (\psi_i + \theta_i) \qquad \ldots\ldots\ldots\ldots\ldots \qquad (2.15)$$

Different approaches are attempted as follows:

a) (2.14) and (2.15) are converted to two iterative formulae and σ and τ are calculated simultaneously. μ is fixed and known ($\mu = 0$) .

$$\sigma_{k+1} = \frac{\tau_k}{n} \sum_{i=1}^{n} (\psi_i + \theta_i) \qquad \ldots\ldots\ldots\ldots \qquad (2.16)$$

$$\tau_k = 3(M_2 - \sigma_k^{\ 2}) \qquad \ldots\ldots\ldots\ldots \qquad (2.17)$$

b) Substitute (2.14) into the original likelihood function i.e.

$$L(\mu, \tau, \sigma) = L(\mu, \tau(\sigma), \sigma) = L^*(\mu, \sigma) \qquad \ldots\ldots \qquad (2.18)$$

and μ, σ are estimated by maximizing the function L^* . The corresponding iteration formula is

$$\begin{pmatrix} \mu_{k+1} \\ \\ \sigma_{k+1} \end{pmatrix} = \begin{pmatrix} \mu_k \\ \\ \sigma_k \end{pmatrix} - \begin{pmatrix} \frac{\partial P}{\partial \mu} & \frac{\partial P}{\partial \sigma} \\ \\ \frac{\partial Q}{\partial \mu} & \frac{\partial Q}{\partial \sigma} \end{pmatrix}^{-1} \begin{pmatrix} P \\ \\ Q \end{pmatrix} \qquad \ldots\ldots\ldots \qquad (2.19)$$

where $P = \frac{\partial \ln L^*}{\partial \mu}$, $Q = \frac{\partial \ln L^*}{\partial \sigma}$, all the functions are evaluated at μ_k, σ_k .

Estimate of τ is obtained from formula (2.14).

2.3.3. 'Correct' Likelihood

The 'correct' likelihood function is

$$L_c(z_1, \ldots, z_n | \mu, \tau, \sigma) = K \prod_i p_i^{f_i}$$

where K is constant and f_i is the number of observations z_j recorded as equal to $i\Delta$, where Δ is a fixed constant, and

$$p_i = \frac{1}{2\tau} \int_a^b [\Phi(\frac{z-\mu+\tau}{\sigma}) - \Phi(\frac{z-\mu-\tau}{\sigma})] \, dz \qquad \ldots\ldots\ldots\ldots \quad (2.20)$$

where $b = (i + \frac{1}{2})\Delta$ and $a = (i - \frac{1}{2})\Delta$. For the sake of convenience, let $p_i' = 2\tau p_i$, then

$$\ln L_c = \ln K + \sum_i f_i \ln p_i \qquad \ldots\ldots\ldots\ldots \quad (2.21)$$

$$\frac{\partial \ln L_c}{\partial \mu} = \sum_i f_i \frac{1}{p_i} \frac{\partial p_i}{\partial \mu}$$

$$= \sum_i \frac{f_i}{p_i'} [\Phi(\frac{z-\mu-\tau}{\sigma}) - \Phi(\frac{z-\mu+\tau}{\sigma})] \Big|_{(i-\frac{1}{2})\Delta}^{(i+\frac{1}{2})\Delta} \qquad \ldots\ldots \quad (2.22)$$

$$\frac{\partial \ln L_c}{\partial \tau} = \sum_i \frac{f_i}{p_i} \frac{\partial p_i}{\partial \tau}$$

$$= \sum_i \frac{f_i}{p_i'} \frac{\partial p_i'}{\partial \tau} - n$$

$$= \sum_i \frac{f_i}{p_i'} [\Phi(\frac{z-\mu+\tau}{\sigma}) + \Phi(\frac{z-\mu-\tau}{\sigma})] \Big|_{(i-\frac{1}{2})\Delta}^{(i+\frac{1}{2})\Delta} \qquad \ldots\ldots \quad (2.23)$$

$$\frac{\partial \ln L_c}{\partial \sigma} = \sum_i \frac{f_i}{p_i} \frac{\partial p_i}{\partial \sigma}$$

$$= \sum_i \frac{f_i}{p_i'} \left[\phi\left(\frac{z-\mu+\tau}{\sigma}\right) - \phi\left(\frac{z-\mu-\tau}{\sigma}\right)\right]\Bigg|_{(i-\frac{1}{2})\Delta}^{(i+\frac{1}{2})\Delta} \quad \ldots\ldots \quad (2.24)$$

Owing to the computational difficulty of the function $\int_a^b \Phi(k)dk$, numerical results using iteration formulae (2.22), (2.23), (2.24) are not provided here.

§2.4. Discussion on iteration formula

From the simulation results of §5.1, it is found that equation (2.11) is satisfactory, i.e., the iterative procedure usually ended in less than 20 iterations for all cases considered. The iteration in equation (2.10) converges quickly, the procedure usually ended in less than 10 iterations. The estimate μ is not sensitive to the initial guess of τ and σ. The iteration in equation (2.12) converges slowly. Maximization of $L^*(\mu, \sigma)$ is carried out by (2.19). The convergence of this iterative procedure depends on the choice of initial values μ_o, σ_o.

Chapter 3. Method of Moments

In this chapter, we shall first consider the parametric estimation problem of a random variable which is the sum of two independent non-identically distributed random variables. Some basic identifiability theorems were proposed by Sclove who also suggested the method of moments as a general method for attacking such a problem. The above method is applied to our problem, the random variable Z, and is followed by some discussions on simulation results.

§3.1 Introduction

We observed a sum of two random variables $Z = X + Y$, where X and Y are independent, non-identically distributed and individually not observed. We shall assume X has a distribution in the family $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$, and Y has a distribution in the family $\mathcal{Q} = \{Q_\omega : \omega \in \Omega\}$, where Θ and Ω are, respectively, subsets of Euclidean r-space and Euclidean s-space. Thus the family of distributions for the observed random variable Z is the family of convolution distribution $= \{F_{\theta,\omega} = P_\theta * Q_\omega \in \Theta \times \Omega\}$, where $*$ is the convolution operator on distribution functions.

§3.2 Identifiability under Convolution and the method of moments

Before examining the question of how one estimates the parameters θ and ω, it is necessary to examine the question of identifiability: is the mapping from the parameter space $\Theta \times \Omega$ to the family of distributions $\mathcal{F}$ one to one? We shall assume throughout that the families $\mathcal{P}$ and $\mathcal{Q}$ are already identifiable, that is, $P_\theta(x) = P_{\theta'}(x)$ for all x implies $\theta = \theta'$ and

$Q_\omega(y) = Q_{\omega'}(y)$ for all y implies $\omega = \omega'$. We now state the following:

Definition: The family $\mathcal{F} = \{F_{\theta,\omega} : F_{\theta,\omega} = P_\theta * Q_\omega \text{ , } (\theta , \omega) \in \Theta \times \Omega\}$ is said to be identifiable under convolution if $F_{\theta,\omega}(Z) = F_{\theta',\omega'}(Z)$ for all Z (or $F_{\theta,\omega} = F_{\theta',\omega'}$) implies $\theta = \theta'$, $\omega = \omega'$.

The importance of the identifiability question is obviously raised by the problem considered herein, where we observe a random variable $Z = X + Y$, X, Y independent, and we wish to estimate the parameters of the distributions of X and Y. If one has no identifiability under convolution, there is no hope of solving such an estimation problem.

The following theorem gives sufficient conditions for identifiability under convolution.

<u>Theorem 1</u>. Basic Identifiability Theorem

Let Θ and Ω be subsets of Euclidean r-space and s-space respectively and let $\mathcal{F} = \{F_{\theta,\omega} = P_\theta * P_\omega : (\theta , \omega) \in \Theta \times \Omega\}$. Let $Z = (Z_1, \ldots, Z_n)$ be n independent identically distributed random variables with distribution belonging to the class $\mathcal{F}$. Assume that:

(i) there exist real-valued measurable functions $H_1(Z), \ldots, H_{r+s}(Z)$ on Euclidean n-space $\{Z = (Z_1, \ldots, Z_n)\}$ such that

$$h_i(\theta , \omega) = E_{\theta,\omega}\{H_i(Z)\} = \int H_i(Z_1, \ldots, Z_n) dF_{\theta,\omega}(Z_1), \ldots, dF_{\theta,\omega}(Z_n)$$

exists for $i = 1, \ldots, r+s$ for all $F_{\theta,\omega} \in \mathcal{F}$, and

(ii) the transformation from $\Theta \times \Omega$ onto the set $S = \{(h_1, \ldots, h_{r+s}) : h_i = h_i(\theta, \omega), (\theta, \omega) \in \Theta \times \Omega\}$ is one-to-one.

Then, $\mathfrak{F}$ is identifiable under convolution.

Proof: $F_{\theta,\omega} = F_{\theta',\omega'}$ implies $h_i(\theta, \omega) = h_i(\theta', \omega')$ for $i = 1, 2, \ldots, r+s$. Hence, since the transformation from $\Theta \times \Omega$ onto S is one-to-one, its unique inverse exists, which implies $\theta = \theta', \omega = \omega'$.

The following theorem gives a specific solution to the identifiability problem under convolution in the case where $r = s = 1$.

Theorem 2. Let Θ and Ω be open subsets of the real lines and

$$\mathfrak{F} = \{F_{\theta,\omega} = P_\theta * P_\omega : (\theta, \omega) \in \Theta \times \Omega\} .$$

Let $Z = (Z_1, \ldots, Z_n)$ be n independent identically distributed random variables with distribution belonging to $\mathfrak{F}$. Let

(i) there exist real-valued measurable functions $H_1(Z)$ and $H_2(Z)$ on Euclidean n-space $\{Z = (Z_1, \ldots, Z_n)\}$ such that $h_i(\theta, \omega) = E_{\theta,\omega}\{H_i(Z)\}$ exist for $i = 1, 2$ and all $F_{\theta,\omega} \in \mathfrak{F}$ and

(ii) $$\left.\begin{aligned} h_1(\theta, \omega) &= f_1(\theta) \\ h_2(\theta, \omega) &= f_2(\theta) + f_3(\omega) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (3.1)$$

where f_1 and f_2 are continuously differentiable on Θ and f_1 is strictly monotone on Θ, f_3 is continuously differentiable and strictly monotone on Ω.

Then $\mathcal{F}$ is identifiable and the unique inverse of the system of equations in (3.1) is given by

$$\theta = f_1^{-1}(h_1) \qquad \omega = f_3^{-1}\{h_2 - f_2(\theta)\} \quad \ldots\ldots \quad (3.2)$$

where f_1^{-1} and f_3^{-1} are the inverse functions to f_1 and f_3. Estimates for θ and ω are obtained from (3.2) as follows

$$\theta = f_1^{-1}\{H(Z)\} \qquad \omega = f_3^{-1}\{H_2(Z) - f_2(\theta)\} \quad \ldots\ldots \quad (3.3)$$

Proof: The Jacobian of the transformation $(\theta,\ \omega) \to (h_1,\ h_2)$ is given by

$$\begin{vmatrix} \frac{\partial h_1}{\partial\theta} & \frac{\partial h_1}{\partial\omega} \\ \frac{\partial h_2}{\partial\theta} & \frac{\partial h_2}{\partial\omega} \end{vmatrix} = \begin{vmatrix} f_1'(\theta) & 0 \\ f_2'(\theta) & f_3'(\omega) \end{vmatrix} = f_1'(\theta)\, f_3'(\omega)\ .$$

Since by assumption this is non-zero on $\Theta \times \Omega$, the system of equation (3.1) is locally invertible by the inverse function theorem. Furthermore, since $f_1(\theta)$ and $f_3(\omega)$ are strictly monotone on Θ and Ω respectively, we have the inverse functions f_1^{-1} and f_3^{-1} existing and the system of equations (3.1) has the unique inverse given by (3.2). $\mathcal{F}$ is indentifiable by Theorem 1.

The indentifiability theorem given above are quite useful in that they suggest a method of getting estimates of θ and ω , namely, the method of moments. In general, letting

$$\left.\begin{aligned}
H_1(Z) &= \bar{Z} = \frac{1}{n} \sum_{i=1}^{n} Z_i \\
H_2(Z) &= M_2 = \frac{1}{n-1} \sum_{i=1}^{n} (Z_i - \bar{Z})^2 \\
H_3(Z) &= M_3 = \frac{n}{(n-1)(n-2)} \sum_{i=1}^{n} (Z_i - \bar{Z})^3 \\
H_4(Z) &= M_4 = \frac{n^2 - 2n + 3}{(n-1)(n-2)(n-3)} \sum_{i=1}^{n} (Z_i - \bar{Z})^4 \\
&\qquad - \frac{3(2n-3)}{n(n-1)(n-2)(n-3)} \left\{\sum_{i=1}^{n} (Z_i - \bar{Z})^2\right\}^2
\end{aligned}\right\} \quad \ldots\ldots \quad (3.4)$$

(see Crámer, 1945, p.352), we obtain the $r + s$ equation required in the theorems if the required moments exist and, provided these equations are invertible, one can construct moment estimators for $\theta = (\theta_1, \ldots, \theta_r)$ and $\omega = (\omega_1, \ldots, \omega_s)$.

Instead of taking $H_i(Z) = M_i$, $i = 1, \ldots, r + s$, above as unbiased estimators of the central moments of Z to obtain the $r + s$ equations, one may prefer to use the $H_i(Z) = k_i$, $i = 1, \ldots, r + s$, where the k_i are the Fisher k-statistics of the sample $(Z_1, \ldots, Z_n)$, which are unbiased estimators of cumulants of the distribution of Z (see Kendall and Stuart, vol. 1, 1958, p.280-281).

§3.3 Application

We now apply the previous results to our estimation problem.

Let μ_r be the r^{th} central moment of Z

$$E(Z) = \mu \ , \quad \mu_2 = E[(Z - \mu)^2] = \sigma^2 + \frac{\tau^2}{3} \quad \ldots\ldots \quad (3.5)$$

and

$$\mu_4 = E[(Z-\mu)^4] = \frac{\tau^4}{5} + 2\tau^2\sigma^2 + 3\sigma^4 \quad \ldots\ldots\ldots \quad (3.6)$$

let

$$M_2 = \frac{n}{n-1} m_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.7)$$

$$M_4 = \frac{n(n^2-2n+3)}{(n-1)(n-2)(n-3)} m_4 - \frac{3n(2n-3)}{(n-1)(n-2)(n-3)} m_2^2 \quad \ldots\ldots \quad (3.8)$$

where

$$m_2 = \frac{1}{n}\sum_{i=1}^{n}(Z_i - \overline{Z})^2$$

$$m_4 = \frac{1}{n}\sum_{i=1}^{n}(Z_i - \overline{Z})^4 .$$

It is a well-known result that M_2 is a consistent and unbiased estimator of 2^{nd}-central moment of Z and M_4 is a consistent and unbiased estimator of 4^{th}-central moment of Z . Also, having neglected terms of order $n^{-\frac{1}{2}}$, we have

$$\mathrm{var}(M_r) = \frac{1}{n}(\mu_{2r} - \mu_r^2 + r^2\mu_2\mu_{r-1}^2 - 2r\,\mu_{r-1}\,\mu_{r+1}) .$$

For the sake of simplicity, let

$$S = \hat{\sigma}^2 \qquad T = \hat{\tau}^2 \quad \ldots\ldots\ldots\ldots\ldots \quad (3.9)$$

Then

$$M_2 = S + \frac{T}{3} \quad \ldots\ldots\ldots\ldots\ldots \quad (3.10)$$

$$M_4 = \frac{T^2}{5} + 2TS + 3S^2 \quad \ldots\ldots\ldots\ldots\ldots \quad (3.11)$$

which give

$$S = M_2 - \sqrt{\frac{5}{2} M_2^2 - \frac{5}{6} M_4} \quad \ldots\ldots\ldots\ldots \quad (3.12)$$

$$T = 3 \sqrt{\frac{5}{2} M_2^2 - \frac{5}{6} M_4} \quad \ldots\ldots\ldots\ldots \quad (3.13)$$

Let m_r' be the r^{th} sample orude moment of Z , i.e.,

$$m_r' = \frac{1}{n} \sum_{i=1}^{n} Z_i^r \quad \ldots\ldots\ldots\ldots \quad (3.14)$$

We express M_2 and M_4 in terms of m_r'

$$M_2 = \frac{m_2' - \frac{m_1'^2}{n}}{n - 1} \quad \ldots\ldots\ldots\ldots \quad (3.15)$$

$$M_4 = m_4' - 4m_4'm_1' + \frac{3(2n-1)}{(n+1)(n-1)} (m_2')^2$$

$$- \frac{6(n-1)}{(n+1)} m_1'^2 m_1'^2 - \frac{2n(n-2)}{(n+1)(n-1)} m_1'^4 \quad \ldots \quad (3.16)$$

Hence, $\hat{\mu}, \hat{\tau}, \hat{\sigma}$ can be expressed in terms of M_1, M_2, M_4 . And the estimates of μ, τ, σ are obtained.

Chapter 4. Estimation involving sample characteristic function

In this chapter, we shall introduce several methods of parametric estimation proposed by Press (1972). All the methods involve sample characteristic functions. Some of them will be applied to our problem. Several simulation results of some iterative formulae is provided §5.3, §5.5.

§4.1 Introduction

Let n independent observations $z_1, \ldots, z_n$ be taken. The problem is to estimate μ, τ, σ^2. Consistent estimators are developed by using sample characteristic functions of the observations.

Denote the sample characteristic function of the observations by

$$f^*(t) = \frac{1}{n} \sum_{j=1}^{n} e^{itz_j} \qquad \ldots\ldots\ldots\ldots \qquad (4.1)$$

Thus, $f^*(t)$ is computable for all values of t. Note that $\{f^*(t), -\infty < t < +\infty\}$ is a stochastic process, and for each t, $|f^*(t)|$ is bounded above by unity. Hence, all moments of $f^*(t)$ are finite, and $f^*(t)$, for any fixed t is the sample average of independent and identically distributed random variables. Thus, by the law of large numbers, $f^*(t)$ is a consistent estimator of $\hat{f}(t)$.

§4.2 General Theory

4.2.1 Estimation Method I (Minimum Distance)

Define

$$g(\mu, \tau, \sigma^2) \equiv \sup_t |f^*(t) - \hat{f}(t)| \quad \ldots\ldots\ldots\ldots \quad (4.2)$$

Where the minimum distance estimators of (μ, τ, σ^2) are the values of these parameters which minimize $g(\mu, \tau, \sigma^2)$. It is well known that these estimators are strongly consistent.

4.2.2 Estimation Method II (Minimum r^{th}-Mean Distance)

Define

$$h(\mu, \tau, \sigma^2) \equiv \int_{-\infty}^{+\infty} |f^*(t) - \hat{f}(t)|^r w(t)dt \quad \ldots\ldots\ldots \quad (4.3)$$

Where $w(t)$ denotes a suitable convergence factor such as $w(t) \equiv (2\pi)^{-\frac{1}{2}} \exp(-\frac{t^2}{2})$, or $w(t) \equiv e^{-|t|}$. Then the minimum r^{th} mean distance estimators are those values of (μ, τ, σ^2) which minimize $h(\mu, \tau, \sigma^2)$ for a fixed r, $r \geqslant 1$. It is not clear which r should be selected, or which weight function $w(t)$ is most appropriate. However, consistent estimators are obtained in any case. Numerous other norms of $|f^*(t) - \hat{f}(t)|$ may be used and it is not clear how to select the optimal one.

4.2.3 Estimation Method III (Moment Estimation)

In what follows an estimation procedure is discussed which yields explicit estimators and involves minimal computation. This procedure is a version of the method of moments.

From (4.1)

$$|\hat{f}(t)| = \left|\frac{\sin t\tau}{t\tau}\right| e^{-\frac{\sigma^2 t^2}{2}} \quad \ldots\ldots\ldots\ldots \quad (4.4)$$

hence

$$\log|\hat{f}(t)| = \log\left|\frac{\sin t\tau}{t\tau}\right| - \frac{\sigma^2 t^2}{2} \quad \ldots\ldots\ldots \quad (4.5)$$

Now choose two nonzero values of t, say, t_1 and t_2, $t_1 \neq t_2$ then

$$\begin{aligned} \log\left|\frac{\sin t_1\tau}{t_1\tau}\right| - \frac{\sigma^2 t_1^{\,2}}{2} &= \log|\hat{f}(t_1)| \\ \log\left|\frac{\sin t_2\tau}{t_1\tau}\right| - \frac{\sigma^2 t_2^{\,2}}{2} &= \log|\hat{f}(t_2)| \end{aligned} \quad \ldots\ldots\ldots \quad (4.6)$$

Solving these two equations simultaneously for τ and σ, and replacing $\hat{f}(t)$ by its estimated value, give

$$\frac{1}{t_1^{\,2}}\log\left|\frac{\sin t_1\tau}{t_1\tau}\right| - \frac{1}{t_2^{\,2}}\log\left|\frac{\sin t_2\tau}{t_2\tau}\right|$$

$$= \frac{1}{t_1^{\,2}}\log|f^*(t_1)| - \frac{1}{t_2^{\,2}}\log|f^*(t_2)| \quad \ldots\ldots \quad (4.7)$$

To estimate μ, define $u(t) \equiv \mathrm{Im}[\log \hat{f}(t)]$, where $\mathrm{Im}[\psi(t)]$ denotes the imaginary part of any complex valued function $\psi(t)$. Then from (4.1)

$$u(t) = \mu t$$

If we choose one nonzero value of t, say t_o, then

$$\mu = \frac{u(t_o)}{t_o} \quad \ldots\ldots\ldots\ldots\ldots \quad (4.8)$$

since $f^*(t) = (\frac{1}{n}\sum_{j=1}^{n} \cos tz_j) + i(\frac{1}{n}\sum_{j=1}^{n} \sin tz_j)$ in polar coordinates,

$f^*(t) \equiv \rho(t) \exp[i\,\theta(t)]$, where

$$\rho^2(t) = (\frac{1}{n}\sum_{j=1}^{n} \cos tz_j)^2 + (\frac{1}{n}\sum_{j=1}^{n} \sin tz_j)^2$$

and

$$\tan\theta(t) = \frac{(\sum_{j=1}^{n} \sin tz_j)}{(\sum_{j=1}^{n} \cos tz_j)}$$

Hence $\log f^*(t) = \rho(t) + i\,\theta(t)$

$$u(t) = \mathrm{Im}\,[\log f^*(t)] = \theta(t)$$

Choose the principal values of $\log f^*(t_o)$, that is, use principal values for $t = t_o$. Replacing $u(t)$ in (4.9) by its estimated value gives

$$\mu = \frac{u(t_o)}{t_o} = \arctan\left(\frac{\sum_{j=1}^{n} \sin t_o z_j}{\sum_{j=1}^{n} \cos t_o z_j}\right) \Big/ t_o \qquad \ldots\ldots \quad (4.9)$$

§4.3 The Estimation Procedure

We use the Estimation Method II in Section 1 to estimate μ, τ, σ^2 respectively. Choose $r = 2$, and $w(t) \equiv e^{\frac{-t^2}{2}}$, let

$$\Xi^* = \min_{\mu,\tau,\sigma^2} h(\mu, \tau, \sigma^2) \equiv \min_{\mu,\tau,\sigma^2} \int_{-\infty}^{+\infty} |\hat{f}(t) - f^*(t)|^2 e^{-t^2} dt \quad \ldots\ldots \quad (4.10)$$

we then put (4.10) in a form more suitable for computation. Define

$$\lambda(t) \equiv |\hat{f}(t) - f^*(t)|$$

hence

$$\lambda(t) \equiv \left(\frac{1}{n}\sum_{\nu=1}^{n} \cos tz_v - \frac{\sin t\tau \cos t\mu}{t\tau} e^{-\frac{\sigma^2 t^2}{2}}\right)^2$$

$$+ \left(\frac{1}{n}\sum_{\nu=1}^{n} \cos tz_v - \frac{\sin t\tau \sin t\mu}{t\tau} e^{-\frac{\sigma^2 t^2}{2}}\right)^2 \quad \ldots\ldots \quad (4.11)$$

The integration in (4.10) is effected numerically by 20 point Hermitian quadrature as

$$\int_{-\infty}^{+\infty} \lambda(t)e^{-t^2}dt \equiv \sum_{k=1}^{20} w_k\lambda(t_k) \quad \ldots\ldots\ldots\ldots \quad (4.12)$$

Where the u_k are the zeros of the Hermite polynomials of degree 20 and w_k are the weights associated with these zeros (Abramowitz and Stegun, 1964, p.924).

We have chosen $w(t) = e^{-t^2}$ since it forces convergence of the integral in (4.11) because of the computational advantage associated with Hermitian quadrature; we have chosen $r = 2$ since it has a greater degree of mathematical tractability than other values.

Now

$$I^* = \min_{\mu,\tau,\sigma^2} \sum_{K=1}^{20} w_k \lambda(t_k)$$

let $\quad C_k = \frac{1}{n}\sum_{\nu=1}^{n} \cos t_k z_v$

$\quad S_k = \frac{1}{n}\sum_{\nu=1}^{n} \sin t_k z_v$

let $\quad P = \sum_{K=1}^{20} w_k \lambda(t_k)$

$$\therefore P = \sum_{K=1}^{20} w_k \Big(C_k - \frac{\sin t_k\tau \cos t_k\mu}{t_k\tau} e^{-\frac{\sigma^2 t_k^2}{2}}\Big)^2$$

$$+ \sum_{K=1}^{20} w_k \Big(S_k - \frac{\sin t_k\tau \cos t_k\mu}{t_k\tau} e^{-\frac{\sigma^2 t_k^2}{2}}\Big)^2 \quad \ldots\ldots \quad (4.13)$$

$$\frac{\partial P}{\partial \mu} = \frac{2}{\tau}\sum_{K=1}^{20} w_k (C_k \sin t_k\mu - S_k \cos t_k\mu) \sin t_k\tau\, e^{-\frac{\sigma^2 t_k^2}{2}} \quad \ldots\ldots \quad (4.14)$$

$$\frac{\partial P}{\partial \sigma^2} = \sum_{K=1}^{20} w_k t_k (C_k \cos t_k\mu + S_k \sin t_k\mu) \frac{\sin t_k\tau}{\tau} e^{-\frac{\sigma^2 t_k^2}{2}}$$

$$- \sum_{K=1}^{20} w_k \Big(\frac{\sin t_k\tau}{\tau}\Big)^2 e^{-\sigma^2 t_k^2} \quad \ldots\ldots \quad (4.15)$$

$$\frac{\partial P}{\partial \tau} = -\frac{1}{\tau}\sum_{K=1}^{20} w_k\, e^{-\frac{\sigma^2 t_k^2}{2}} \Big(\cos t_k\tau - \frac{\sin t_k\tau}{t_k\tau}\Big) \cdot$$

$$\cdot \Big(C_k \cos t_k\mu + S_k \sin t_k\mu - \frac{\sin t_k\tau}{t_k\tau} e^{-\frac{\sigma^2 t_k^2}{2}}\Big) \ldots \quad (4.16)$$

let $\quad \frac{\partial P}{\partial \mu} = F(\mu) \qquad \frac{\partial P}{\partial \sigma^2} = G(\sigma^2) \qquad \frac{\partial P}{\partial \tau} = H(\tau)$

we want to solve the equations

$$F(\mu) = G(\sigma^2) = H(\tau) = 0$$

by applying Newton-Raphson method separately, the iteration formula are

$$\mu_{k+1} = \mu_k - \frac{F(\mu_k)}{F'(\mu_k)} \quad \ldots\ldots\ldots\ldots \quad (4.17)$$

$$\sigma^2_{k+1} = \sigma^2_k - \frac{G(\sigma^2_k)}{G'(\sigma^2_k)} \quad \ldots\ldots\ldots\ldots \quad (4.18)$$

$$\tau_{k+1} = \tau_k - \frac{H(\tau_k)}{H'(\tau_k)} \quad \ldots\ldots\ldots\ldots \quad (4.19)$$

where

$$F'(\mu) = \frac{2}{\tau}\sum_{k=1}^{20} w_k t_k (S_k \sin t_k\mu + C_k \cos t_k \mu) \sin t_k \tau \, e^{-\frac{\sigma^2 t_k^2}{2}} \quad \ldots \quad (4.20)$$

$$G'(\sigma^2) = -\sum_{k=1}^{20} \frac{w_k t_k^3}{2} (C_k \cos t_k\mu + S_k \sin t_k \mu) \sin \frac{t_k \tau}{2} \, e^{-\frac{\sigma^2 t_k^2}{2}}$$

$$+ \sum_{k=1}^{20} w_k t_k^2 \left(\frac{\sin t_k \tau}{\tau}\right)^2 e^{-\sigma^2 t_k^2} \quad \ldots\ldots\ldots \quad (?.21)$$

$$H'(\tau) = -\frac{1}{\tau}\sum_{k=1}^{20} w_k e^{-\frac{\sigma^2 t_k^2}{2}} \left[-\left(\frac{t_k\tau \cos t_k \tau - \sin t_k \tau}{t_k \tau^2}\right)^2 e^{-\frac{\sigma^2 t_k^2}{2}} \right.$$

$$+ \left(C_k \cos t_k \mu + S_k \sin t_k \mu - \frac{\sin t_k \mu}{t_k \mu} e^{-\frac{\sigma^2 t_k^2}{2}}\right) \cdot$$

$$\left. \cdot \left(\frac{\sin t_k \tau (2 - t_k^2 \tau^2)}{t_k \tau^3} - 2\frac{\cos t_k \tau}{\tau^2}\right)\right] \quad \ldots . \quad (4.22)$$

On the other hand, the equation

$$F(\mu, \tau, \sigma^2) = G(\mu, \tau, \sigma^2) = H(\mu, \tau, \sigma^2) = 0$$

can be solved by the following iteration formula in matrix form

$$\begin{pmatrix} \mu_{k+1} \\ \sigma^2_{k+1} \\ \tau_{k+1} \end{pmatrix} = \begin{pmatrix} \mu_k \\ \sigma^2_k \\ \tau_k \end{pmatrix} - \begin{pmatrix} \frac{\partial F}{\partial \mu} & \frac{\partial F}{\partial \sigma^2} & \frac{\partial F}{\partial \tau} \\ \frac{\partial G}{\partial \mu} & \frac{\partial G}{\partial \sigma^2} & \frac{\partial G}{\partial \tau} \\ \frac{\partial H}{\partial \mu} & \frac{\partial H}{\partial \sigma^2} & \frac{\partial H}{\partial \tau} \end{pmatrix}^{-1} \begin{pmatrix} F \\ G \\ H \end{pmatrix} \quad \ldots\ldots \quad (4.23)$$

where the value of the function involved are evaluated at $\mu = \mu_k$, $\tau = \tau_k$ and $\sigma^2 = \sigma_k^2$ respectively.

§4.4 Discussion of iteration procedure

Except equation (4.19) which converges extremely slowly, (each improvement is approximately of the order 10^{-8}, i.e., $|\tau_i - \tau_{i+1}| \sim 10^{-8}$), the iteration procedure enbodied converges in less than 20 iterations for all cases considered. The behaviour of equation (4.18) is comparatively stable. For wide range of initial values of μ, τ, σ^2 , it converges almost to the same final value.

Equation (4.18) is sensitive to the initial value of τ. It is possible that the integral in (4.17, 4.18) does not possess a global minimum although we do not know whether this integral is a convex function of μ, τ, σ^2 or not. Convexity, of course, would ensure a global minimum. The moment estimators are usually ineffective.

Chapter 5. Monte Carlo Studies of Estimates

To determine the extent to which the formulae derived in previous chapters can be used to estimate μ, τ and σ the simulation of Z is necessary since a general mathematical analysis of the procedures seems to be quite difficult. Sample values of Z are simulated as follow: Two sequences $\{\xi_i\}$ and $\{\eta_i\}$, $i = 1, \ldots, n$ of uniform random variates are generated in an HP9830. By suitable transformations $\{\xi_i\}$ is converted to $\{X_i\}$ and $\{\eta_i\}$ is converted to $\{Y_i\}$. Direct addition gives the sample values of Z, i.e. Z_i, $i = 1, \ldots, n$.

Throughout this chapter, n is defined as the sample size and r is the number of repetitions.

§5.1 Simulation results of the combined method

In chapter 2, the combined method was derived from method of maximum likelihood and method of moments. Using the iterative formulae (2.16) and (2.17) the simulation results are as follows.

$n = 50$ $\qquad$ $r = 100$ $\qquad$ $\varepsilon = 0.0001$

a) $\mu = 0$ $\qquad$ $\tau = 4$ $\qquad$ $\sigma = 1$

Average of $\hat{\tau}$	s.d. of $\hat{\tau}$	Average of $\hat{\sigma}$	s.d. of $\hat{\sigma}$
3.62947	1.49703	0.94311	0.67010

b)

$\mu = 0$	$\tau = 3$	$\sigma = 2$	
Average of $\hat{\tau}$	s.d. of $\hat{\tau}$	Average of $\hat{\sigma}$	s.d. of $\hat{\sigma}$
3.21353	1.65476	1.89137	0.93716

c)

$\mu = 0$	$\tau = 2$	$\sigma = 3$	
Average of $\hat{\tau}$	s.d. of $\hat{\tau}$	Average of $\hat{\sigma}$	s.d. of $\hat{\sigma}$
2.84028	2.14759	2.37147	1.37145

τ and σ are estimated values of τ and σ respectively.

We observe that the degree of precision of estimates decreases as the ratio $\tau/_{\sigma}$ decreases. For the case $\tau = 4, \sigma = 1$, we find that among 100 repetitions, 12 estimated τ values cluster around 0.15 i.e., the average of them is 0.15547 and standard devation is 0.02338. The remaining estimates cluster around 4, the true value of τ, with average 4.10823 and standard deviation 0.19724.

§5.2. Simulation results of method of moments

In chapter 3, estimators of μ, τ, σ are derived from the method of moments. The simulation results of the unbiased estimator of mean and variance of Z, i.e., $\overline{Z}$ and M_2 respectively are as follows.

a)

$n = 100$,	$r = 100$,	$\mu = 2$,	$\tau = 2$,	$\sigma = 0.4$
Average of $\overline{Z}$	s.d. of $\overline{Z}$	Average of M_2	s.d. of M_2	
1.94533	0.12499	1.49141	0.13057	

b) $n = 100$, $r = 100$, $\mu = 2$, $\tau = 2$, $\sigma = 2$

Average of $\bar{Z}$	s.d. of $\bar{Z}$	Average of M_2	s.d. of M_2
1.93034	0.13989	4.86595	0.23975

c) $n = 100$, $r = 100$, $\mu = 2$, $\tau = 2$, $\sigma = 3.6$

Average of $\bar{Z}$	s.d. of $\bar{Z}$	Average of M_2	s.d. of M_2
2.2111	0.19222	14.20581	0.32072

The estimates are stable and with satisfactory precision in general. On the other hand, the simulation results of M_4 of equation (3.8) together with the 4^{th} sample central moment are as follows.

$r = 50$, $\mu = 1.5$, $\tau = 1$, $\sigma = 1$

n	Average of M_4	s.d. of M_4	Average of m_4	s.d. of m_4
100	5.12807	1.75288	5.03255	1.71196
1000	5.15444	0.55413	5.14443	0.55280

The average of M_4 is quite close to the 4^{th} central moment μ_4, which is equal to 5.2. As the sample size increases, the 4^{th} sample central moment is close to the 4^{th} central moment also. The estimates of τ and σ using equations (3.12, (3.13) are 0.76249 and 1.06359 with standard deviations 0.13849 and 0.08535 respectively.

§5.3. Simulation results of method of sample characteristic function

The simulation results of (4.17), (4.18) are as follows.

$n = 50$ $r = 50$ $\varepsilon = 0.0001$

a) $\mu = 1$ $\tau = 1$ $\sigma = 1$

Average of $\hat{\mu}$	s.d. of $\hat{\mu}$	Average of $\hat{\sigma}$	s.d. of $\hat{\sigma}$
0.98342	0.11893	0.99788	0.10555

b) $\mu = 1$ $\tau = 2$ $\sigma = 1$

Average of $\hat{\mu}$	s.d. of $\hat{\mu}$	Average of $\hat{\sigma}$	s.d. of $\hat{\sigma}$
0.96556	0.18590	0.97500	0.20955

c) $\mu = 1$ $\tau = 3$ $\sigma = 1$

Average of $\hat{\mu}$	s.d. of $\hat{\mu}$	Average of $\hat{\sigma}$	s.d. of $\hat{\sigma}$
0.98263	0.16401	1.02964	0.26889

d) $\mu = 1$ $\tau = 4$ $\sigma = 1$

Average of $\hat{\mu}$	s.d. of $\hat{\mu}$	Average of $\hat{\sigma}$	s.d. of $\hat{\sigma}$
0.99174	0.27003	1.04177	0.33860

e) $\mu = 1$ $\tau = 4$ $\sigma = 2$

Average of $\hat{\mu}$	s.d. of $\hat{\mu}$	Average of $\hat{\sigma}$	s.d. of $\hat{\sigma}$
0.96886	0.17546	1.94587	0.39930

f) $\mu = 1$ $\tau = 4$ $\sigma = 3$

Average of $\hat{\mu}$	s.d. of $\hat{\mu}$	Average of $\hat{\sigma}$	s.d. of $\hat{\sigma}$
0.88352	0.33801	2.88984	0.32836

$n = 50 \quad r = 50 \quad \varepsilon = 0.0001$

g) $\mu = 1 \quad \tau = 4 \quad \sigma = 4$

Average of μ	s.d. of μ	Average of σ	s.d. of σ
1.02777	0.27442	3.87758	0.34721

The values of τ are substituted by its true values.

We observe that the degree of precision of estimates of μ is stable. The degree of precision of estimates of σ decreases as τ increases.

§5.4 Simulation result of estimate of μ by maximum likelihood

In chapter 2, estimator of μ is derived from the method of maximum likelihood. In the iterative formula (2.10) value of τ and σ are fixed, not necessary equal to the true value. The simulation results are as follows.

$n = 100 \quad r = 50 \quad \varepsilon = 0.0001$

$\mu = 2 \quad \tau = 2 \quad \sigma = 2$

	Assume $\tau = 2$ $\sigma = 2$	Assume $\tau = 3$ $\sigma = 3$	Assume $\tau = 4$ $\sigma = 4$
Average of μ	2.01307	2.01269	2.12871
s.d. of μ	0.17198	0.17132	0.17855

From the above result, we observe that the accuracy of estimate of μ does not change with respect to the assumed value of τ and σ.

§5.5 Comparison of estimates of μ, τ, σ

a) Comparison of estimates of μ .

Among the three methods i.e. method of maximum likelihood, method of moments and method of sample characteristic function, the standard deviations of μ are of order 10^{-1} .

b) Comparison of estimates of τ and σ

Between the combined method and method of moment, the combined method is more applicable even if the sample size is small. On the other hand, the method of moment is not always applicable if the sample size is small, i.e., we cannot take square root of negative number (see equation (3.12) and (3.13)). In comparing the standard deviations, the method of moment has greater precision. Finally, the failure of estimation of τ using iteration formulae (2.12) and (4.19) do not imply the failure of estimation of τ by maximum likelihood and method of sample characteristic function. As we will see in the next section, formula (2.19) and (4.23) lead to successful estimate of τ .

c) comparison of simultaneous estimates of μ, τ, σ .

Among the combined method, method of moments and the method involving sample characteristic function, a comparison is carried out by using equation (2.19), (4.23) and the method of moment for $\mu = 0.5$, $\tau = 0.5$ and $\sigma = 0.1$. In the following table, let

I - the method involving sample characteristic function

II - the combined method

III - the method of moment

r - the number of repetitions

n - the sample size

i) comparison of estimates of μ

$\mu = 0.5$, $r = 100$

		I	II	III
n = 50	mean	0.49805	0.49926	0.50316
	M.S.E.	1.8543×10^{-3}	1.9182×10^{-3}	1.5010×10^{-3}
n = 100	mean	0.49878	0.50385	0.49193
	M.S.E.	9.5710×10^{-4}	9.2756×10^{-4}	7.8294×10^{-4}
n = 150	mean	0.49922	0.49846	0.50295
	M.S.E.	6.9408×10^{-4}	6.5380×10^{-4}	5.2482×10^{-4}

ii) comparison of estimates of τ

$\tau = 0.5$, $r = 100$

		I	II	III
n = 50	mean	0.48530	0.49395	0.49193
	M.S.E.	2.2401×10^{-3}	1.6040×10^{-3}	4.4757×10^{-3}
n = 100	mean	0.49390	0.50476	0.49193
	M.S.E.	9.8657×10^{-4}	8.6255×10^{-4}	1.5978×10^{-3}
n = 150	mean	0.49772	0.49396	0.49743
	M.S.E.	1.0200×10^{-4}	6.2751×10^{-4}	1.6236×10^{-3}

iii) comparison of estimates of σ

$\sigma = 0.1$, $r = 100$

		I	II	III
n = 50	mean	0.10036	0.094555	0.12708
	M.S.E.	7.0464×10^{-3}	1.4944×10^{-3}	2.0764×10^{-3}
n = 100	mean	0.10127	0.093157	0.11129
	M.S.E.	5.6523×10^{-3}	1.2277×10^{-3}	1.0481×10^{-3}
n = 150	mean	0.098700	0.097104	0.10646
	M.S.E.	5.2056×10^{-3}	7.27767×10^{-4}	8.2974×10^{-4}

The following table denotes the number of trials required to give 100 estimates.

	I	II	III
N = 50	147	170	104
N = 100	137	137	100
N = 150	115	122	100

In comparing the estimates of μ among the three methods, method of moment is the most stable method and the most accurate method as the other two methods is not always successful.

In comparing the estimates of τ among the three methods, the combined method is the better one when the sample size is small and the method involving sample characteristic function is better when the sample size is large.

In comparing the estimates of σ among the three methods, the combined method is the better one when the sample size is small or the sample size is large. The degree of precision of the combined method is improving as sample size increases.

Both the combined method and the method involving sample characteristic function are less stable than the method of moments in comparing their number of trials needed to give 100 estimates and the method involving sample characteristic function is least stable. For the case that the sample size is 50 the number of trials needed for 100 successful results is 170.

References

1. Barnard, G.A. (1967). The use of the likelihood function in statistical practice. 5th Berkeley Symp., 1, 27-40.

2. Bhattacharjee, G.P., et al. (1963). Dimensional chains involving rectangular and normal error-distributions, Technometrics, 5, 404-406.

3. Cramer, H. (1945). Mathematical Methods of Statistics, Princeton: University Press, 346-352.

4. Fisher, R.A.(1921). On the mathematical foundations of theoretical statistics. Phil. Trans. Roy. Soc. London, A, 222, 309-368.

5. Giesbrecht, F. and Kempthorne, O.(1970). Maximum likelihood estimation in the three-parameter lognormal distribution. J. Roy. Statist. Soc., B. 38, 257-264.

6. Kempthorne, O. (1966). Some aspects of experimental inforence. J. Amer. Statist. Ass., 61, 11-34.

7. Kendall, M.G. and Stuart, A. (1958). The Advanced Theory of Statistics, vol.I. London: Griffin. P.

8. Kulldorff, G. (1957). On the conditions for consistency and asymptotic efficiency of maximum likelihood estimates. Skaud. Aktuar., 40, 129-144.

9. Lambert, J.A. (1970). Estimation of parameters in the four-parameter lognormal. Autral. J. Statist., 12, 33-44.

10. Paulson, A.S., et al. (1975). The estimation of the parameters of the stable laws. Biometrika, 62, 163-169.

11. Press, S.J. (1972). Estimation in univariate and multivariate stable distributions. J. Amer. Statist. Ass., 67, 842-846.

12. Sclove, S.L., and Van Ryzin, J. (1969). Estimating the parameters of a convolution: J. Roy. Statist. Soc. B, 31, 181-191.